Euler's formula for zeta(2n) and Cauchy variables

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EULER'S FORMULAE FOR $\zeta(2n)$ AND CAUCHY VARIABLES

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Abstract. Euler's formulae for $\zeta(2n)$ are recovered from the computation in two different manners of the even moments of $\log(|\mathbb{C}_1\mathbb{C}_2|)$, for $\mathbb{C}_1$ and $\mathbb{C}_2$ two independent standard Cauchy variables.
The method employed is generalized first to L_{χ_4} and then to other trigonometric series.

1. Introduction

1.1. Consider the series developments, for $t = 1$ and 2,

$$\frac{1}{(\cos(\theta))^t} = \sum_{n=0}^{\infty} \frac{A_n^{(t)}}{(2n)!}\,\theta^{2n} \quad \left(|\theta| < \frac{\pi}{2}\right).$$

The coefficients $\left(A_n^{(1)}, n \geq 0\right)$ and $\left(A_n^{(2)}, n \geq 0\right)$ are well known to be $A_n^{(1)} = A_{2n}$ and $A_n^{(2)} = A_{2n+1}$, respectively the Euler or secant numbers, and the tangent numbers (more information about A_{2n} and A_{2n+1} can be found in [5]).
On the other hand, consider both the zeta function

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} \quad (\Re s > 1)$$

and the L function associated with the quadratic character χ_4 :

$$L_{\chi_4}(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^s} \quad (\Re s > 0)\,.$$

The following formulae are very classical (see for example [7]) :

$$L_{\chi_4}(2n+1) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1} \frac{A_n^{(1)}}{\Gamma(2n+1)}, \tag{1}$$

$$\left(1 - \frac{1}{2^{2n+2}}\right)\zeta(2n+2) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2} \frac{A_n^{(2)}}{\Gamma(2n+2)}. \tag{2}$$

1.2. In this note, we show that formulae (1) and (2) may be obtained simply by computing the moments $\mathbb{E}\left((\Lambda_1)^{2n}\right)$ and $\mathbb{E}\left((\Lambda_2)^{2n}\right)$, where $\Lambda_1 = \log\left(|\mathbb{C}_1|\right)$ and $\Lambda_2 = \log\left(|\mathbb{C}_1\mathbb{C}_2|\right)$, with $\mathbb{C}_1$ and $\mathbb{C}_2$ two independent standard Cauchy variables.

- On one hand these moments can be computed explicitly in terms of L_{χ_4} and ζ respectively, thanks to explicit formulae for the densities of Λ_1 and Λ_2.
- On the other hand, these moments may be obtained via the representation

$$|\mathbb{C}_1| \overset{\text{law}}{=} e^{\frac{\pi}{2}\hat{C}_1}, \tag{3}$$

where $\hat{C}_1$ is a random variable whose distribution is characterized by

$$\mathbb{E}\left(e^{i\lambda\hat{C}_1}\right) = \frac{1}{\cosh\lambda} \quad (\lambda \in \mathbb{R})$$

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or

$$\mathbb{E}\left(e^{\theta \hat{C}_1}\right) = \frac{1}{\cos\theta} \quad \left(|\theta| < \frac{\pi}{2}\right). \tag{4}$$

More precisions about $\hat{C}_1$ or even the Levy process $(\hat{C}_t, t \geq 0)$ can be found in [5].

1.3. In the next two parts, the moments of Λ_1 and Λ_2 are calculated according to the method explained above. Then we give a probabilistic interpretation of equality (3), which is a key step in our demonstration.

Finally, section 5 presents a generalization of these computations, using the same method, applied to the ratio of two stable variables.

1.4. The most popular ways to prove (1) and (2) make use of Fourier inversion and Parseval's theorem, or of non trivial expansions of functions such as cotan (see for example [7]).

The method we present is elementary, except for the equality (3), for which we give a analytic proof in 3.2, and a probabilistic one in section 4.

2. The even moments of Λ_1 and Λ_2

As is well known, the density of $\mathbb{C}_1$ is

$$\delta_1(x) = \frac{1}{\pi(1+x^2)}.$$

It is not difficult to show that δ_2, the density of $\mathbb{C}_1\mathbb{C}_2$, is

$$\delta_2(x) = \frac{2\log|x|}{\pi^2(x^2-1)}.$$

From the knowledge of δ_1 and δ_2 we deduce the following result.

Proposition 1. *The even moments of* Λ_1 *and* Λ_2 *are given by*

$$\mathbb{E}\left[(\Lambda_1)^{2n}\right] = \frac{4}{\pi}\Gamma(2n+1)L_{\chi_4}(2n+1), \tag{5}$$

$$\mathbb{E}\left[(\Lambda_2)^{2n}\right] = \frac{8}{\pi^2}\Gamma(2n+2)\left(1 - \frac{1}{2^{2n+2}}\right)\zeta(2n+2). \tag{6}$$

Proof. The LHS of (5) equals

$$\frac{2}{\pi}\int_0^\infty \frac{(\log x)^{2n}\mathrm{d}x}{1+x^2} = \frac{4}{\pi}\int_1^\infty \frac{(\log x)^{2n}\mathrm{d}x}{1+x^2}.$$

Then, making the change of variables $x = e^u$, followed by the series expansion $\frac{1}{1+e^{-2u}} = \sum_{k=0}^\infty (-1)^k e^{-2ku}$, we obtain formula (5).

The proof of formula (6) relies on the same argument, starting from the expression of δ_2. □

3. Obtention of formulae (1) and (2)

3.1. Let us assume formula (3), and define a variable $\hat{C}_2$ such that

$$e^{\frac{\pi}{2}\hat{C}_2} \overset{\text{law}}{=} |\mathbb{C}_1\mathbb{C}_2|.$$

We note that

$$\hat{C}_1 \overset{\text{law}}{=} \frac{2}{\pi}\log|\mathbb{C}_1| \overset{\text{law}}{=} \frac{2}{\pi}\Lambda_1$$

and likewise

$$\hat{C}_2 \overset{\text{law}}{=} \frac{2}{\pi}\Lambda_2.$$

Then the even moments of $\hat{C}_1$ and $\hat{C}_2$ are given by

$$\mathbb{E}\left[(\hat{C}_t)^{2n}\right] = A_n^{(t)} \quad (t = 1, 2)$$

so that, from the relations between $\hat{C}_t$ and Λ_t, we get

$$\mathbb{E}\left[(\Lambda_t)^{2n}\right] = \left(\frac{\pi}{2}\right)^{2n} A_n^{(t)} \quad (t = 1, 2). \tag{7}$$

Putting together formulae (7)-(8) on one hand, and formula (9) on the other hand, we obtain the desired results (1) and (2).

3.2. To finish completely our proof, it now remains to show formula (3), that is, starting with $\mathbb{C}_1$, to show that

$$\mathbb{E}\left[e^{i\lambda \frac{2}{\pi}\log|\mathbb{C}_1|}\right] = \frac{1}{\cosh\lambda} \quad (\lambda \in \mathbb{R}). \tag{8}$$

The LHS of (8) is $\mathbb{E}\left[|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}}\right]$. To compute this quantity we use the fact that $\mathbb{C}_1 \stackrel{\text{law}}{=} N/N'$, where N and N' are two standard independent Gaussian variables. We shall also use the fact that $N^2 \stackrel{\text{law}}{=} 2\gamma_{1/2}$ where γ_a is a gamma(a) variable. Thus, we have

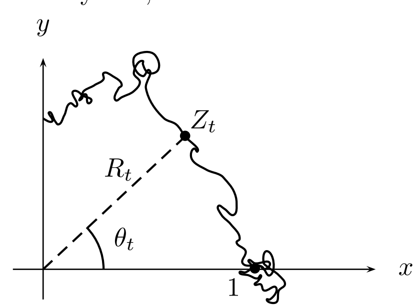
$$\mathbb{E}\left[|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}}\right] = \left|\mathbb{E}\left[(\gamma_{1/2})^{\frac{i\lambda}{\pi}}\right]\right|^2 = \frac{\left|\Gamma\left(\frac{1}{2} + i\frac{\lambda}{\pi}\right)\right|^2}{\left(\Gamma(\frac{1}{2})\right)^2} = \frac{1}{\cosh(\lambda)} \quad (\lambda \in \mathbb{R}).$$

For a proof of this last identity see [3], Problem 1 p. 14.

4. Understanding the relation (3) in terms of planar Brownian motion

Since our derivation of the study of the identity (8) is rather analytical, it seems of interest to provide a more probabilistic proof of it.

4.1. Consider $Z_t = X_t + iY_t$ a $\mathbb{C}$−valued Brownian motion, starting from $1 + i0$. Denote $R_t = |Z_t| = (X_t^2 + Y_t^2)^{1/2}$, and $(\theta_t, t \geq 0)$ a continuous determination of the argument of $(Z_u, u \leq t)$ around 0, with $\theta_0 = 0$.



Recall that there exist two independent one-dimensional Brownian motions $(\beta_u, u \geq 0)$ and $(\gamma_u, u \geq 0)$ such that

$$\log R_t = \beta_{H_t}, \text{ and } \theta_t = \gamma_{H_t}. \tag{9}$$

Next, we consider $T = \inf\{t : X_t = 0\} = \inf\left\{t : |\theta_t| = \frac{\pi}{2}\right\}$.

4.2. Now, from (9) we obtain, on the one hand,

$$H_T = \inf\left\{u : |\gamma_u| = \frac{\pi}{2}\right\} \stackrel{\text{def}}{=} T_{\pi/2}^{\gamma,*},$$

and, on the other hand, it is well known that Y_T is distributed as $\mathbb{C}_1$; therefore, using (9), we obtain $\log|\mathbb{C}_1| \stackrel{\text{law}}{=} \beta_{T_{\pi/2}^{\gamma,*}}$, so that

$$\frac{2}{\pi}\log|\mathbb{C}_1| \stackrel{\text{law}}{=} \beta_{T_1^{\gamma,*}}.$$

Consequently, thanks to the independence of β and γ, we obtain

$$\mathbb{E}\left[e^{i\lambda\frac{2}{\pi}\log|\mathbb{C}_1|}\right] = \mathbb{E}\left[e^{i\lambda\beta_{T_1^{\gamma,*}}}\right] = \mathbb{E}\left[e^{-\frac{\lambda^2}{2}T_1^{\gamma,*}}\right] = \frac{1}{\cosh\lambda},$$

as is well known.

4.3. More details and applications to the asymptotic study of jumps of the Cauchy process are provided in [6].

In a similar vein, the reader will find some closely related computations by P. Levy [4] who, for the same purpose as ours, uses Fourier inversion of the characteristic functions $1/\cosh\lambda$, $\lambda\sinh\lambda$ and $1/(\cosh\lambda)^2$.

Appendix : a slight generalization

The formulae. Let $X_\mu = \frac{T_\mu}{T'_\mu}$, with T_μ and T'_μ two independent, unilateral, stable variables with exponent μ :

$$\mathbb{E}\left[e^{-\lambda T_\mu}\right] = e^{-\lambda^\mu}.$$

Although, except for $\mu = 1/2$, the density of T_μ does not admit a simple expression, we know from Lamperti [2] (see also Chaumont-Yor [1] exercise 4.21) that

$$\mathbb{E}\left[(X_\mu)^s\right] = \frac{\sin \pi s}{\mu \sin\left(\frac{\pi s}{\mu}\right)}, \tag{10}$$

$$\mathbb{P}\left((X_\mu)^\mu \in \mathrm{d}y\right) = \frac{\sin(\pi\mu)}{\pi\mu}\frac{\mathrm{d}y}{y^2 + 2y\cos(\pi\mu)+1}. \tag{11}$$

As in the previous sections, we calculate $\mathbb{E}\left[(\log X_\mu^\mu)^{2n}\right]$ in two different ways.

- If we define the sequence $(a_n^{(\mu)}, n \geq 0)$ via the Taylor series $\frac{\sin\pi s}{\mu\sin\left(\frac{\pi s}{\mu}\right)} = \sum_{n\geq 0}\frac{a_n^{(\mu)}}{(2n)!}(\pi s)^{2n}$ then, from (10),
$$\mathbb{E}\left[(\log(X_\mu)^\mu)^{2n}\right] = \pi^{2n}a_n^{(\mu)}. \tag{12}$$
- We rewrite (11) as $\mathbb{P}\left((X_\mu)^\mu \in \mathrm{d}y\right) = \frac{\mathrm{d}y}{2i\pi\mu}\left(\frac{1}{y+e^{-i\pi\mu}} - \frac{1}{y+e^{i\pi\mu}}\right)$. With the usual series expansion we get
$$\mathbb{E}\left[(\log(X_\mu)^\mu)^{2n}\right] = \frac{2\Gamma(2n+1)}{\pi\mu}\sum_{k\geq 1}\frac{(-1)^{k+1}\sin(k\mu\pi)}{k^{2n+1}}. \tag{13}$$

Formulae (12) and (13) give

$$\sum_{k\geq 1}\frac{(-1)^{k+1}\sin(k\mu\pi)}{k^{2n+1}} = \frac{\pi^{2n+1}\mu}{2\Gamma(2n+1)}\ a_n^{(\mu)}. \tag{14}$$

Comments about (14).

- Formula (14) with $\mu = 1/2$ gives $L_{\chi_4}(2n+1) = \frac{\pi^{2n+1}}{4\Gamma(2n+1)}\ a_n^{(1/2)}$, which is coherent with formula (1).
- Formula (2) about ζ can also be generalized via the random variable X_μ. We consider now the product of two independent variables X_μ and $\tilde{X}_\mu$. We then need to introduce the Taylor expansion of $\left(\frac{\sin\pi s}{\mu\sin\left(\frac{\pi s}{\mu}\right)}\right)^2$ and the density of $(X_\mu)^\mu(\tilde{X}_\mu)^\mu$, which is
$$\mathbb{P}\left((X_\mu)^\mu(\tilde{X}_\mu)^\mu \in \mathrm{d}y\right) = \frac{\mathrm{d}y}{(2\pi\mu)^2}\left(\frac{-\log y - 2i\pi\mu}{y-e^{-2i\pi\mu}} + \frac{-\log y + 2i\pi\mu}{y - e^{2i\pi\mu}} + \frac{2\log y}{y-1}\right).$$
The straightforward calculations for $\mathbb{E}\left[(\log((X_\mu)^\mu(\tilde{X}_\mu)^\mu))^{2n}\right]$ are left to the reader.

- Formula (14) looks like the famous formula

$$\sum_{k=0}^{\infty} \frac{\sin((2k+1)\mu\pi)}{(2k+1)^{2n+1}} = \frac{(-1)^n \pi^{2n+1}}{4(2n)!} E_{2n}(\mu), \tag{15}$$

where E_{2n} is the $2n^{th}$ Euler polynomial. Formula (14) (with μ replaced by 2μ) and (15) together give an explicit expression (for all $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$) for

$$\sum_{k\geq 1} \frac{\sin(k\mu\pi)}{k^{2n+1}}. \tag{16}$$

The derivative of (16) with respect to μ gives an explicit expression for

$$\sum_{k\geq 1} \frac{\cos(k\mu\pi)}{k^{2n}}.$$

For $\mu = 0$, we get the expression for $\zeta(2n)$. To summarize, we have found two ways to prove formula (2) : the one which uses the product of two Cauchy variables, and a second one which uses the one parameter family (X_μ) generalizing the Cauchy variable.

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